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A general method to calculate all irreducible representations of a finite group

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Abstract. A so-called stochastic method is introduced which can be used to decompose the reducible representations of any finite group into irreducible representations. In this way, the regular representation of a finite group is reduced completely to obtain all of the irreducible representations of the group. It also induces a new approach to calculate the character table of a finite group. Based on this method, these calculations or decompositions can be performed by means of a computer.

1. Introduction

As is well known, there are two fundamental problems in the theory of representation of finite groups. Firstly, finding all irreducible finite-dimensional representations of a given finite group; and secondly, decomposing a given finite-dimensional representation of a finite group into its irreducible components. It is evident that the first problem is more important than the second one because we can decompose the reducible representations of a finite group completely if the irreducible representations of the group are known (Elliott and Dawber 1979). However, the first problem has not been completely solved yet, although some developments have been made (Cannon 1969, Flodmark and Blokker 1967, 1972, Flodmark and Jansson 1982, Bradly and Cracknell 1972, Chen 1981, 1982, 1984, Davies 1982, Neubuser 1982, Folland 1977, 1979, Chen *et al* 1985).

In terms of the theory of representation of finite groups, given the generators and the relations between the generators or the group table of a finite group, of which the order g is known, we can easily get the conjugate classes $C_i (i = 1, 2, \ldots, r)$, the number r of classes and the regular representation $\Gamma^{\mathrm{reg}}_A (\forall A \in \mathrm{G})$. The regular representation Γ^{reg} is a unitary reducible representation and contains all irreducible representations of the given group. For our purposes, we only discuss how to decompose the regular representation of a non-Abelian group into the irreducible representations as follows. However, the method given in the present paper can also be used to decompose the other reducible representations.

Let r be the number of classes of a given group G. There certainly exist r non-equivalent irreducible representations $\Gamma^{(\alpha)} (\alpha = 1, 2, \ldots, r)$ and each of them appears d_α times in the regular representation, where d_α is the dimension of the irreducible representations $\Gamma^{(\alpha)}$. Therefore, the regular representation of the group G

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may be written in the following form:

$$\Gamma_A^{\text{reg}} = U \begin{pmatrix} \Gamma_A^{(1)} & & & & & & & \\ & \ddots & & & & & 0 & \\ & & \Gamma_A^{(1)} & & & & & \\ & & & \Gamma_A^{(2)} & & & & \\ & & & & \ddots & & & \\ & & & & & \Gamma_A^{(r)} & & \\ & 0 & & & & & \ddots & \\ & & & & & & & \Gamma_A^{(r)} \end{pmatrix} U^{+} \qquad (\forall A \in \mathrm{G}) \tag{1.1}$$

where U is a unitary matrix.

If the matrix U can be found, the regular representation of the group G can be decomposed and all the irreducible representations of the group G can be extracted from the decomposition of a regular representation of G. Unfortunately, we know little about the matrix U except that it is unitary and transforms the regular representation Γ^{reg} into the form (1.1), so it is difficult to find U. The main aim of the present paper is to introduce the general method used to decompose the regular representation and to find the matrix U in equation (1.1), i.e. we find a method to answer the first problem mentioned above.

There exists a theorem in the theory of linear algebra as follows (Zhang 1980).

Theorem 1. If matrices A and B commute with each other, i.e.

$$AB = BA$$

and the matrix A also has the form

$$A = \begin{pmatrix} \lambda_1 I_1 & & & 0 \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ 0 & & & \lambda_r I_r \end{pmatrix} \tag{1.2}$$

where $\lambda_i \neq \lambda_j$ while $i \neq j$, then B has the form

$$B = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_r \end{pmatrix} \tag{1.3}$$

where B_i is a d_i-dimensional matrix with its unit matrix I_i $(i = 1, 2, \ldots, r)$. On the other hand, if A can be transformed into the form (1.2) by a unitary matrix, B can also be kept in the form (1.3) by the same unitary matrix.

From theorem 1, we obtain a very important conclusion as follows.

If there is a Hermitian matrix X which commutes with all the matrices Γ_A^i of a unitary reducible representation Γ^i of a group G, i.e.

$$\Gamma_A^i X = X \Gamma_A^i \qquad (\forall A \in \mathrm{G}) \tag{1.4}$$

and if the eigenvalues of the matrix X are not equal to the same number, then a unitary matrix making X diagonal can reduce Γ^i to the direct sum of the representations of G with dimension less than the dimension of Γ^i.

Using this conclusion, we can proceed in the following way.

(i) Find the matrix X in Γ^{reg}, which satisfies the conditions

$$X^{\dagger} = X \qquad \Gamma_A^{\text{reg}} X = X \Gamma_A^{\text{reg}} \qquad (\forall A \in \mathrm{G}) \tag{1.5}$$

and then calculate the unitary matrix V making X diagonal, by which we reduce Γ^{reg} into r reducible representations $(\Gamma)_\alpha$, i.e.

$$V^{\dagger}\Gamma_A^{\text{reg}} V = \begin{pmatrix} (\Gamma_A)_1 & & & \\ & (\Gamma_A)_2 & & \\ & & \ddots & \\ & & & (\Gamma_A)_r \end{pmatrix} \qquad (\forall A \in \mathrm{G}). \tag{1.6}$$

Here $(\Gamma)_\alpha$ is equivalent to the direct sum of the irreducible representations $\Gamma^{(\alpha)}$, with the number of irreducible $\Gamma^{(\alpha)}$ contained in $(\Gamma)_\alpha$ being d_α in terms of group theory, i.e.

$$(\Gamma_A)_\alpha = W_\alpha \begin{pmatrix} \Gamma_A^{(\alpha)} & & & \\ & \Gamma_A^{(\alpha)} & & \\ & & \ddots & \\ & & & \Gamma_A^{(\alpha)} \end{pmatrix} W_\alpha^{\dagger} \qquad \forall A \in \mathrm{G} \tag{1.7}$$

where $W_\alpha^{\dagger} = W_\alpha^{-1}$.

(ii) Find matrices Z_α in $(\Gamma)_\alpha (\alpha = 1, 2, \ldots, r)$ which satisfy the conditions

$$Z_\alpha^{\dagger} = Z_\alpha \qquad (\Gamma_A)_\alpha Z_\alpha = Z_\alpha (\Gamma_A)_\alpha \qquad (\forall A \in \mathrm{G}) \tag{1.8}$$

respectively. We require Z_α to have at least two non-equal eigenvalues, so that the matrix diagonalising Z_α may reduce the reducible representations $(\Gamma)_\alpha$ further until the reducible representations are completely reduced.

The fundamental problem now becomes how to find X and $Z_\alpha (\alpha = 1, 2, \ldots, r)$.

2. The initial decomposition of the regular representation

In this section, we discuss how to find a Hermitian matrix X in equation (1.5) for Γ^{reg}. Let S_i^{reg} be the sum of the matrices of the elements in the class $C_i (i = 1, 2, \ldots, r)$ which belongs to the regular representation Γ^{reg}, i.e.

$$S_i^{\text{reg}} = \sum_{A \in C_i} \Gamma_A^{\text{reg}}$$

and also

$$S_{i^{-1}}^{\text{reg}} = \sum_{A \in C_i} \Gamma_{A^{-1}}^{\text{reg}}$$

where A^{-1} stands for the inverse element of $A (\forall A \in \mathrm{G})$. In terms of the properties of the conjugate class C_i and the regular representation Γ^{reg} (Lomont 1959), the relations are

$$S_i^{\text{reg}} \Gamma_A^{\text{reg}} = \Gamma_A^{\text{reg}} S_i^{\text{reg}} \qquad \forall A \in \mathrm{G} \qquad i = 1, 2, \ldots, r. \tag{2.1a}$$

$$S_{i^{-1}}^{\text{reg}} = (S_i^{\text{reg}})^{\dagger} \tag{2.1b}$$

Construct X in the form

$$X = \sum_{i=1}^{r} p_i S_i^{\text{reg}} \tag{2.2}$$

where p_i are stochastic complex numbers. It is easy to confirm that X commutes with $\Gamma_A^{\mathrm{reg}}(\forall A \in \mathrm{G})$ due to the property (2.1a). In order to make X Hermitian, we require that

$$p_{i^{-1}} = p_i^* \qquad i = 1, 2, \ldots, r \tag{2.3}$$

(if $C_{i^{-1}} = C_i$, p_i is real). Therefore, X with constraint (2.3) must satisfy conditions (1.5).

We need to prove that the number of distinct eigenvalues of X are at most r and the multiplicities of the distinct eigenvalues are $d_1^2, d_2^2, \ldots, d_r^2$, respectively, as long as X really has r distinct eigenvalues.

From a theorem in group theory (Miller 1972, Naimark 1980), the decomposition of the regular representation can always be written as in equation (1.1), without knowing the form of U, so we have

$$U^\dagger X U = \sum_{i=1}^{r} p_i \left(\sum_{A \in C_i} U^\dagger \Gamma_A^{\mathrm{reg}} U \right)$$

$$= \sum_{i=1}^{r} \sum_{A \in C_i} p_i \begin{pmatrix} \Gamma_A^{(1)} & & & & & & \\ & \ddots & & & & & \\ & & \Gamma_A^{(1)} & & & & \\ & & & \Gamma_A^{(2)} & & & \\ & & & & \ddots & & \\ & & & & & \Gamma_A^{(r)} & \\ & & & & & & \ddots & \\ & & & & & & & \Gamma_A^{(r)} \end{pmatrix} . \tag{2.4}$$

As

$$\left(\sum_{A \in C_i} \Gamma_A^{(\alpha)} \right) \Gamma_B^{(\alpha)} = \Gamma_B^{(\alpha)} \left(\sum_{A \in C_i} \Gamma_A^{(\alpha)} \right) \qquad (\forall B \in \mathrm{G}) \tag{2.5}$$

according to Schur's lemma, which states that a matrix commuting with all the matrices of an irreducible representation must be a numerical matrix, we have

$$\sum_{A \in C_i} \Gamma_A^{(\alpha)} = a_i^{(\alpha)} I_\alpha \qquad \alpha, i = 1, 2, \ldots, r \tag{2.6}$$

where $a_i^{(\alpha)}$ is an undetermined constant and I_α is a d_α-dimensional unit matrix. Taking the trace on both sides of equation (2.6), we obtain

$$a_i^{(\alpha)} = \frac{1}{d_\alpha} \mathrm{Tr} \left(\sum_{A \in C_i} \Gamma_A^{(\alpha)} \right) = \frac{g_i}{d_\alpha} \chi_i^{(\alpha)} \tag{2.7}$$

where g_i is the number of elements in C_i and $\chi_i^{(\alpha)}$ is a character of a class C_i for the irreducible representation $\Gamma^{(\alpha)}$. Substituting (2.5) and (2.6) into equation (2.4), we have

$$U^\dagger X U = \begin{pmatrix} q_1 I_1 & & & \\ & q_2 I_2 & & \\ & & \ddots & \\ & & & q_r I_r \end{pmatrix} \tag{2.8}$$

where

$$q_\alpha = \sum_{i=1}^{r} p_i g_i \left(\frac{\chi_i^{(\alpha)}}{d_\alpha} \right) \qquad \alpha = 1, 2, \ldots, r. \tag{2.9}$$

From equation (2.8), it is easy to see that the greatest number of distinct eigenvalues of X is r and their multiplicities are $d_1^2, d_2^2, \ldots, d_r^2$, respectively.

However, we do not expect that X has the r eigenvalues when the values of the stochastic numbers $p_i (i = 1, 2, \ldots, r)$, with constraint (2.3), are chosen at random. We should discuss the probability of X having just r distinct eigenvalues. For any two non-equivalent irreducible representations $\Gamma^{(\alpha)}$ and $\Gamma^{(\beta)}$, the characters $(\chi_1^{(\alpha)}, \chi_2^{(\alpha)}, \ldots, \chi_r^{(\alpha)})$ are not proportional to the characters $(\chi_1^{(\beta)}, \chi_2^{(\beta)}, \ldots, \chi_r^{(\beta)})$, so $(\chi_1^{(\alpha)}/d_\alpha, \chi_2^{(\alpha)}/d_\alpha, \ldots, \chi_r^{(\alpha)}/d_\alpha)$ is not equal to $(\chi_1^{(\beta)}/d_\beta, \chi_2^{(\beta)}/d_\beta, \ldots, \chi_r^{(\beta)}/d_\beta)$ at all. If the stochastic $p_1, p_2, \ldots, p_r$ satisfy one or more of the following conditions:

$$\sum_{i=1}^{r} p_i g_i \chi_i^{(\alpha)}/d_\alpha = \sum_{i=1}^{r} p_i g_i \chi_i^{(\beta)}/d_\beta \qquad \alpha, \beta = 1, 2, \ldots, r \qquad \alpha \neq \beta \tag{2.10}$$

then the number of distinct eigenvalues of X will be less than r. Since $(p_1, p_2, \ldots, p_r)$ can take any set of values in the R^r real space and each condition of equation (2.10) represents a hyperplane of $(r-1)$ dimensions in R^r, X has r distinct eigenvalues if the values of $(p_1, p_2, \ldots, p_r)$ are not taken on one or more of the $\frac{1}{2}r(r-1)$ hyperplanes. Therefore, the probability is high when X has r distinct eigenvalues.

Taking the values of the complex stochastic numbers $p_1, p_2, \ldots, p_r$ with constraint (2.3) freely, we form a Hermitian matrix X and calculate the eigenvalues of X. It will have r distinct eigenvalues as usual, or if not, we change the stochastic numbers. If X has r distinct eigenvalues, the multiplicities of them are $d_1^2, d_2^2, \ldots, d_r^2$, respectively, as pointed out before, so that we obtain the dimension of the irreducible representations of G. The eigenvectors of X form a unitary matrix V, which will diagonalise the matrix X and will also decompose the regular representation of G.

The eigenvalues of X usually degenerate corresponding to irreducible representations whose dimension is greater than one. As a result of this, V cannot decompose the regular representation completely, but it can decompose the regular representation partially as in equation (1.6). Thus we have to decompose the reducible representations $(\Gamma)_\alpha$ further in the next step.

As mentioned previously, the complex stochastic numbers $p_1, p_2, \ldots, p_r$ satisfy the constraint (2.3). In order to avoid the necessity of determining whether a class contains the inverse elements of another class and to simplify the form of X, we may write X in the following form. Let

$$p_j = \alpha_j' + \mathrm{i}\beta_j' \qquad (\text{where } \mathrm{i} = \sqrt{-1})$$

$$\alpha_j' = \alpha_j + \alpha_{j-1} \qquad \beta_j' = \beta_j - \beta_{j-1}.$$

α_j, β_j, $\alpha_{j^{-1}}$, $\beta_{j^{-1}}$ are real stochastic numbers and we obtain

$$p_j S_j^{\text{reg}} + p_{j^{-1}} S_{j^{-1}}^{\text{reg}} = p_j S_j^{\text{reg}} + p_j^* (S_j^{\text{reg}})^+$$

$$= \alpha_j' [S_j^{\text{reg}} + (S_j^{\text{reg}})^{\text{T}}] + \mathrm{i}\beta_j' [S_j^{\text{reg}} - (S_j^{\text{reg}})^{\text{T}}]$$

$$X = \sum_{j=1}^{r} \{\alpha_j [S_j^{\text{reg}} + (S_j^{\text{reg}})^{\text{T}}] + \mathrm{i}\beta_j [S_j^{\text{reg}} - (S_j^{\text{reg}})^{\text{T}}]\}. \tag{2.11}$$

3. The final decomposition of the regular representation

In order to decompose the reducible representations $(\Gamma)_\alpha$ further, we need to find a Hermitian matrix Z_α commuting with $(\Gamma_A)_\alpha (\forall A \in \mathrm{G})$ and having at least two distinct eigenvalues. Let Y be an arbitrary Hermitian matrix, and set up

$$Z_\alpha = \sum_{A \in \mathrm{G}} (\Gamma_{A^{-1}})_\alpha Y (\Gamma_A)_\alpha. \tag{3.1}$$

It is easy to confirm that

$$Z_\alpha^\dagger = Z_\alpha \qquad (\Gamma_A)_\alpha Z_\alpha = Z_\alpha (\Gamma_A)_\alpha \qquad (\forall A \in \mathrm{G}). \tag{3.2}$$

For convenience of calculation and discussion, it is supposed that

$$Y = \begin{pmatrix} Y_0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{3.3}$$

Y_0 is a d_α-dimensional Hermitian matrix and is restricted with $\mathrm{Tr}\ Y_0 \neq 0$, and the elements of Y_0 consist of the complex stochastic numbers

$$\begin{aligned} (Y_0)_{mk} &= \omega_{mk} + \mathrm{i}\gamma_{mk} \\ (Y_0)_{km} &= (Y_0)^*_{mk} = \omega_{mk} - \mathrm{i}\gamma_{mk} \end{aligned} \qquad (m, k = 1, 2, \dots, d_\alpha) \tag{3.4}$$

where ω_{mk}, γ_{mk} are real stochastic numbers. First, we will find the conditions for the eigenvalues of Z_α being equal when we discuss the probability that Z_α has at least two distinct eigenvalues.

Assume

$$Q^\dagger Z_\alpha Q = b_\alpha I_{d_\alpha^2} \qquad \text{where } Q^\dagger = Q^{-1} \tag{3.5a}$$

i.e.

$$Z_\alpha = b_\alpha I_{d_\alpha^2} \tag{3.5b}$$

where b_α is an undetermined constant and $I_{d_\alpha^2}$ is a d_α^2-dimensional unit matrix. From equations (3.1)-(3.3), we obtain

$$b_\alpha = \frac{1}{d_\alpha^2} \mathrm{Tr}\ Z_\alpha = \frac{g}{d_\alpha^2} \mathrm{Tr}\ Y_0 \neq 0. \tag{3.6}$$

As is well known, a unitary matrix W_α exists that decomposes $(\Gamma)_\alpha$ into the direct sum of the irreducible representations $\Gamma^{(\alpha)}$, i.e.

$$(\Gamma_A)_\alpha = W_\alpha \begin{pmatrix} \Gamma_A^{(\alpha)} & & & \\ & \Gamma_A^{(\alpha)} & & \\ & & \ddots & \\ & & & \Gamma_A^{(\alpha)} \end{pmatrix} W_\alpha^\dagger \qquad (\forall A \in \mathrm{G}). \tag{3.7}$$

We write W_α and $W_\alpha^\dagger$ in the form

$$W_\alpha = \begin{pmatrix} W_{11} & W_{12} & \dots & W_{1d_\alpha} \\ W_{21} & W_{22} & \dots & W_{2d_\alpha} \\ \dots & \dots & \dots & \dots \\ W_{d_\alpha,1} & W_{d_\alpha,2} & \dots & W_{d_\alpha,d_\alpha} \end{pmatrix} \qquad W_\alpha^\dagger = \begin{pmatrix} W_{11}^\dagger & W_{21}^\dagger & \dots & W_{d_\alpha,1}^\dagger \\ W_{12}^\dagger & W_{22}^\dagger & \dots & W_{d_\alpha,2}^\dagger \\ \dots & \dots & \dots & \dots \\ W_{1d_\alpha}^\dagger & W_{2d_\alpha}^\dagger & \dots & W_{d_\alpha,d_\alpha}^\dagger \end{pmatrix} \tag{3.8}$$

where $W_{ij}(i, j = 1, 2, \dots, d_\alpha)$ are d_α-dimensional submatrices.

Substituting equations (3.5)-(3.8) into equation (3.1), we have

$$W_\alpha^\dagger Z_\alpha W_\alpha = W_\alpha^\dagger \left(\sum_{A \in G} (\Gamma_{A^{-1}})_\alpha Y (\Gamma_A)_\alpha \right) W_\alpha$$

$$= \begin{pmatrix} \sum_{A\in G} \Gamma_{A^{-1}}^{(\alpha)} W_{11}^\dagger Y_0 W_{11} \Gamma_A^{(\alpha)} & \sum_{A\in G} \Gamma_{A^{-1}}^{(\alpha)} W_{11}^\dagger Y_0 W_{12} \Gamma_A^{(\alpha)} & \dots & \sum_{A\in G} \Gamma_{A^{-1}}^{(\alpha)} W_{11}^\dagger Y_0 W_{1d_\alpha} \Gamma_A^{(\alpha)} \\ \sum_{A\in G} \Gamma_{A^{-1}}^{(\alpha)} W_{12}^\dagger Y_0 W_{11} \Gamma_A^{(\alpha)} & \sum_{A\in G} \Gamma_{A}^{(\alpha)} W_{12}^\dagger Y_0 W_{12} \Gamma_A^{(\alpha)} & \dots & \sum_{A\in G} \Gamma_{A^{-1}}^{(\alpha)} W_{12}^\dagger Y_0 W_{1d_\alpha} \Gamma_A^{(\alpha)} \\ \dots & \dots & \dots & \dots \\ \sum_{A\in G} \Gamma_{A^{-1}}^{(\alpha)} W_{1d_\alpha}^\dagger Y_0 W_{11} \Gamma_A^{(\alpha)} & \sum_{A\in G} \Gamma_{A^{-1}}^{(\alpha)} W_{1d_\alpha}^\dagger Y_0 W_{12} \Gamma_A^{(\alpha)} & \dots & \sum_{A\in G} \Gamma_{A^{-1}}^{(\alpha)} W_{1d_\alpha} Y_0 W_{1d_\alpha} \Gamma_A^{(\alpha)} \end{pmatrix}$$

$$= b_\alpha I_{d_\alpha^2}. \tag{3.9}$$

Since there exists the relation between an irreducible representation and a matrix M (Hamermesh 1962)

$$\sum_{A\in G} \Gamma_{A^{-1}}^{(\alpha)} M \Gamma_A^{(\alpha)} = \frac{g \operatorname{Tr} M}{d_\alpha} I_{d_\alpha}$$

then these conditions are obtained from equation (3.6) and equation (3.9):

$$\left.\begin{aligned} &\operatorname{Tr}(W_{1i} W_{1i}^\dagger Y_0) = b_\alpha \frac{d_\alpha}{g} = \frac{1}{d_\alpha} \operatorname{Tr} Y_0 \\ &\operatorname{Tr}(W_{1i} W_{1j}^\dagger Y_0) = 0 \end{aligned}\right. \qquad i \neq j \qquad i, j = 1, 2, \dots, d_\alpha \tag{3.10a}$$

or

$$\left.\begin{aligned} &\operatorname{Tr}\left(\left(W_{1i} W_{1i}^\dagger - \frac{1}{d\alpha} I_{d_\alpha}\right) Y_0\right) = 0 \\ &\operatorname{Tr}(W_{1i} W_{1j}^\dagger Y_0) = 0 \end{aligned}\right. \qquad i \neq j \qquad i, j = 1, 2, \dots, d_\alpha \tag{3.10b}$$

while the eigenvalues of Z_α have the same value. In fact, the matrices $W_{1i} W_{1i}^\dagger - (1/d_\alpha) I_{d_\alpha}$ and $W_{1i} W_{1j}^\dagger (i \neq j,\ i, j = 1, 2, \dots, d_\alpha)$ cannot all be zero. The reason is that if

$$W_{1i} W_{1i}^\dagger - (1/d_\alpha) I_{d_\alpha} = 0 \qquad i = 1, 2, \dots, d_\alpha$$

then

$$\det(W_{1i} W_{1i}^\dagger) = |\det W_{1i}|^2 = \det((1/d_\alpha) I_{d_\alpha}) \neq 0$$

i.e.

$$\det W_{1i} \neq 0 \qquad \det W_{1i}^\dagger \neq 0 \qquad i = 1, 2, \dots, d_\alpha.$$

Thus

$$\det(W_{1i} W_{1j}^\dagger) = (\det W_{1i})(\det W_{1j}^\dagger) \neq 0 \qquad i \neq j \qquad i, j = 1, 2, \dots, d_\alpha.$$

As a result, $W_{1i} W_{1j}^\dagger (i \neq j,\ i, j = 1, 2, \dots, d_\alpha)$ cannot all be zero simultaneously. On the other hand, if

$$W_{1i} W_{1j}^\dagger = 0 \qquad i \neq j \qquad i, j = 1, 2, \dots, d_\alpha$$

then

$$\det W_{1i} \equiv 0 \qquad \text{or} \qquad \det W_{1j}^\dagger \equiv 0 \qquad i \neq j.$$

It is impossible to make

$$W_{1i} W_{1i}^\dagger - (1/d_\alpha) I_{d_\alpha} = 0.$$

So the rank k of the coefficient matrix of the elements of Y_0 in equation (3.10b) is greater than or equal to one, and the solutions of the elements of Y_0 are the hyperplanes of $(d_\alpha^2 - k)$ dimension in the $R^{d_\alpha^2}$ real space. Z_α may become a numerical matrix $b_\alpha I_{d_\alpha^2}$ if the elements of Y_0 are taken on the hyperplanes.

From the above discussion, we know that Z_α has two or more distinct eigenvalues except when the elements of Y_0 are taken on the finite number of hyperplanes (one is due to Tr $Y_0 = 0$, the others as mentioned above). Hence, when the elements of Y_0 are taken at random, the probability that Z_α has two or more distinct eigenvalues is high.

A unitary matrix can be obtained from the eigenvectors of Z_α when Z_α has two or more distinct eigenvalues. By means of the unitary matrix, we can decompose $(\Gamma_A)_\alpha (\forall A \in \mathrm{G})$ into representations whose dimensions are nd_α where n is an integer and $1 \leqslant n < d_\alpha$. If the least dimension of the representations is d_α, the representation with the least dimension is certainly an irreducible representation of G. Otherwise, we again have to construct a Hermitian matrix Z'_α of nd_α dimensions, as above, to calculate continuously until we get the d_α-dimensional irreducible representation.

So far we have seen that the stochastic numbers play a very important role in our method, affecting the matrices X and Z_α. In some cases, the number of the eigenvalues of X or Z_α is less than we require, if we take a set of stochastic numbers for X or Z_α. But fortunately the probability of this always happening is so small that we need not worry about such cases. So, if encountering this case, it is enough to take another set of stochastic numbers to construct the matrices X and Z_α. By means of a computer it is convenient to take a set of stochastic numbers and to calculate the eigenvalues and eigenvectors of X and Z_α.

Moreover, it should be pointed out that this stochastic method is different from those of Flodmark-Blokker and Chen. Flodmark-Blokker's scheme (1972) is rather complicated for constructing the basis of the irreducible representation $\Gamma^{(\alpha)}$ in terms of the diagonal form of some group elements $A(\forall A \in \mathrm{G})$. Chen (1981) has proposed another scheme for selecting a complete set of commuting operators of a group G, but it requires a reconstruction of the subgroup chain of the group G which is more difficult to perform. Comparatively, the stochastic method provides a very useful and efficient method of explicitly carrying out decompositions. Using this method, we have calculated the irreducible representations of all non-Abelian groups of order 81 (Senior and Lunn 1934) which were unknown before. The method, of course, can also be applied to other aspects, for example, to the determination of the character table of finite groups.

4. The method generalised to the calculation of the character table of a finite group†

The stochastic method had been adopted by McKay for many years (see Neubuser 1970) although he only used it to calculate the character table. Here we give more detail and a more complete description. As an application, it will be found that the stochastic method is also effective in calculating the character table.

Among the conjugate classes $C_i (i = 1, 2, \ldots, r)$, the relations

$$C_i C_j = \sum_{k=1}^{r} h_{ijk} C_k \qquad i, j = 1, 2, \ldots, r \tag{4.1}$$

† See Chen and Lee (1985).

exist, where h_{ijk} is an integer or zero. We obtain

$$g_i g_j \chi_i^{(\alpha)} \chi_j^{(\alpha)} = f_\alpha \sum_{k=1}^{r} h_{ijk} g_k \chi_k^{(\alpha)} \qquad \alpha = 1, 2, \ldots, r \qquad i, j = 1, 2, \ldots, r \tag{4.2}$$

from the relation above. The symbols used in equations (4.2) have been defined in the previous sections of this paper. If h_{ijk} is regarded as the element $(H_i)_{jk}$ of a matrix H_i, equations (4.2) can be rewritten as

$$v_i^{(\alpha)} v^{(\alpha)} = H_i v^{(\alpha)} \qquad \alpha, i = 1, 2, \ldots, r \tag{4.3}$$

where $v_i^{(\alpha)} = g_i \chi_i^{(\alpha)} / d_\alpha$ is the component of the eigenvectors $v^{(\alpha)}$ of H_i.

For convenience of discussion, we set

$$l_i^{(\alpha)} = \frac{g_i}{d_\alpha} \chi_i^{(\alpha)} \qquad \varphi_i^{(\alpha)} = \frac{1}{\sqrt{g_i}} v_i^{(\alpha)} = \sqrt{g_i} \chi_i^{(\alpha)} / d_\alpha$$

and also

$$(L_i)_{jk} = \frac{g_k}{(g_j g_k)^{1/2}} h_{ijk}.$$

Equations (4.3) are transformed into

$$(L_i - l_i I)\varphi = 0 \qquad i = 1, 2, \ldots, r \tag{4.4}$$

where I is an r-dimensional unit matrix and l_i is a constant.

According to the properties of h_{ijk}, we have:

(i) corresponding to a class $C_{i^{-1}}$, which contains the inverse elements of a class C_i,

$$L_{i^{-1}} = L_i^{\mathrm{T}} \tag{4.5}$$

where L_i^{T} is the transposed matrix of L_i. Because the elements of L_i^{T} are real, then

$$L_i^{\mathrm{T}} = L_i^{\dagger}. \tag{4.6}$$

(ii) L_i and L_j commute with each other, i.e.

$$L_i L_j = L_j L_i \qquad i, j = 1, 2, \ldots, r. \tag{4.7}$$

We can calculate the eigenvalues of all of equation (4.4) which relate to the characters $x_i^{(\alpha)} (i = 1, 2, \ldots, r)$, but we do not know how to arrange the character table corresponding to the irreducible representations from the eigenvalues. The eigenvectors $\varphi^{(\alpha)} (\alpha = 1, 2, \ldots, r)$ of one of equation (4.4) concerned with the characters $\chi_i^{(\alpha)}$ must be undetermined, since some of the characters for a class are usually equal, which make the eigenvalues degenerate. However, the eigenvectors have the very important property that they are common to all the matrices $L_i (i = 1, 2, \ldots, r)$. Another property they have is that they are orthonormal by pairs due to

$$\sum_{i=1}^{r} g_i \chi_i^{(\alpha)} \chi_i^{(\beta)*} = 0 \qquad \alpha \neq \beta \qquad \alpha, \beta = 1, 2, \ldots, r \tag{4.8}$$

which can be found in any book on the theory of representation of the group. Hence the character table is only given according to a unique orthonormal set of common eigenvectors (to within constant factors which are determined below) of $L_i (i = 1, 2, \ldots, r)$. The key to the problem is how to calculate such eigenvectors.

Similar to theorem 1, another theorem exists.

Theorem 2. If two matrices A and B commute, and if an eigenvalue of A is non-degenerate, the eigenvector associated with it is also an eigenvector of B.

From theorem 2, it is necessary to get a matrix L which commutes with all the matrices $L_i(i=1,2,\ldots,r)$. If all the eigenvalues of L are non-degenerate, the eigenvectors associated with them must be common eigenvectors of all the matrices $L_i(i=1,2,\ldots,r)$. Using the same idea as presented previously, we set

$$L=\sum_{i=1}^{r}\lambda_i L_i \tag{4.9}$$

where $\lambda_i(i=1,2,\ldots,r)$ are a set of stochastic numbers. Similarly, the constraint on the stochastic numbers is

$$\lambda_{i^{-1}}=\lambda_i^*. \tag{4.10}$$

One can prove that

$$L^{+}=L \qquad L_iL=LL_i \qquad (i=1,2,\ldots,r). \tag{4.11}$$

Therefore, equations (4.3) are replaced by the following equation:

$$(L-lI)\varphi=0 \tag{4.12}$$

where $l=\Sigma_{i=1}^{r}\lambda_i l_i$. We determine a unique orthonormal set of common eigenvectors from equation (4.12); if selected correctly the stochastic numbers $\lambda_1,\lambda_2,\ldots,\lambda_r$ make the eigenvalues of L non-degenerate, for the characters in a class corresponding to the non-equivalent irreducible representations are not exactly equal. Hence the eigenvectors associated with them correspond to

$$\begin{aligned}\varphi^{(\alpha)}&=(\varphi_1^{(\alpha)},\varphi_2^{(\alpha)},\ldots,\varphi_r^{(\alpha)})\\&=K^{(\alpha)}\left(\sqrt{g_1}\frac{\chi_1^{(\alpha)}}{d_\alpha},\sqrt{g_2}\frac{\chi_2^{(\alpha)}}{d_\alpha},\ldots,\sqrt{g_r}\frac{\chi_r^{(\alpha)}}{d_\alpha}\right)\end{aligned} \tag{4.13}$$

where $\alpha=1,2,\ldots,r$ for the irreducible representation and $K^{(\alpha)}$ is an undetermined constant.

Consider how to determine the constant factor $K^{(\alpha)}$. Since C_1 contains only one element, the unit element of the group, i.e. $g_1=1$ and $\chi_1^{(\alpha)}=d_\alpha$, we obtain

$$\varphi^{(\alpha)}=\left(1,\sqrt{g_2}\frac{\chi_2^{(\alpha)}}{d_\alpha},\ldots,\sqrt{g_r}\frac{\chi_r^{(\alpha)}}{d_\alpha}\right) \qquad \alpha=1,2,\ldots,r \tag{4.14}$$

normalising $\varphi^{(\alpha)}$ to $\varphi_1^{(\alpha)}=1$. We know the property of characters

$$\sum_{i=1}^{r}g_i|\chi_i^{(\alpha)}|^2=g. \tag{4.15}$$

g is the order of G. So

$$d_\alpha=\left(\frac{g}{\Sigma_{i=1}^{r}|\varphi_i^{(\alpha)}|^2}\right)^{1/2} \qquad \alpha=1,2,\ldots,r. \tag{4.16}$$

The dimensions of the irreducible representations are obtained from equations (4.16). Finally, the characters

$$\chi_i^{(\alpha)}=\frac{d_\alpha}{\sqrt{g_i}}\varphi_i^{(\alpha)} \qquad \alpha,i=1,2,\ldots,r \tag{4.17}$$

are obtained from the eigenvectors and formed into the character table.

As before, we should discuss the probability that the eigenvalues of L are non-degenerate. Using the same idea as in § 2, suppose that if the eigenvalues are degenerate, one or more equations

$$l^{(\alpha)} = l^{(\beta)} \qquad \alpha \neq \beta \qquad \alpha, \beta = 1, 2, \ldots, r \tag{4.18a}$$

or

$$\sum_{i=1}^{r} \lambda_i \frac{g_i \chi_i^{(\alpha)}}{d_\alpha} = \sum_{i=1}^{r} \lambda_i \frac{g_i \chi_i^{(\beta)}}{d_\beta} \qquad \alpha \neq \beta \qquad \alpha, \beta = 1, 2, \ldots, r \tag{4.18b}$$

exist, similar to equation (2.8). We can also prove that the stochastic numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ can be easily chosen to make the eigenvalues of L non-degenerate.

We have proposed a method to determine the character table even though there are some other methods to do so (Chen *et al* 1985, Dixon 1967, Cannon 1969). Comparatively, the advantage of the stochastic method lies in the fact that it only requires equation (4.12) to be treated and is convenient to calculate with a computer. Of course, the method used to decompose the regular representation into the representations $(\Gamma)_\alpha$ in § 2 can also be used to determine the character table, but it is more complicated. Applying the method to physics, it can also be used to construct a basis of eigenvectors common to a complete set of commuting observables which play an important role in quantum mechanics.

5. Concluding remarks

The following correspondences are established for the decomposition of the reducible representations or the determination of the character table of a finite group.

(i) Look for the matrices that commute with element matrices or class matrices of a finite group with conditions (1.5), (1.8) or (4.11).

(ii) Calculate the eigenvalues of these matrices. When the numbers of the distinct eigenvalues are a maximum, the eigenvectors associated with them are composed of unitary matrices which decompose the reducible representations or determine the character table.

The matrices to be found consist of the stochastic numbers which affect the numbers of the eigenvalues. It has been proved that the probability is high when the numbers of the eigenvalues are a maximum and the stochastic numbers are selected.

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